INTRODUCTION TO CYCLIC HOMOLOGY

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Contents

1 Introduction **2**

2 Hochschild (co)homology **2**
2.1 Definition of Hochschild homology . . . 3
2.2 As a derived functor . . . 4
2.3 Definition of Hochschild cohomology . . . 5
2.4 Relation with other (co)homologies . . . 6

3 Cyclic (co)homology **7**
3.1 Definition of cyclic homology . . . 7
3.2 Different points of view . . . 8
3.3 Definition of cyclic cohomology . . . 10
3.4 Connes' exact sequence . . . 11
3.5 Decomposition along conjugacy classes . . . 12

4 Relation with K-theory **13**
4.1 Periodic cyclic homology . . . 13
4.2 Chern character . . . 16

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1 Introduction

Over the past few decades, the language of homological algebra has increasingly become central to the study of algebraic and geometric structures, particularly in contexts where traditional tools of commutative geometry are no longer available. One of the most fruitful developments in this direction has been the emergence of cyclic homology, a theory that arose at the intersection of algebra, topology, and analysis, and has proven to be a key component in the framework of noncommutative geometry.

The origins of cyclic homology lie in the earlier theory of Hochschild homology, introduced to capture the failure of commutativity in associative algebras. Beyond this formal role, Hochschild homology gained a deeper geometric interpretation thanks to the Hochschild–Kostant–Rosenberg theorem, which identifies the Hochschild homology of a smooth commutative algebra with the space of differential forms. This result revealed that the Hochschild complex could serve as a noncommutative model for differential geometry, laying the groundwork for algebraic approaches to smooth manifolds.

Cyclic homology, initially defined by A. Connes in the early 1980's, defines a refinement of the Hochschild homology. It introduces additional structure into the Hochschild complex, either via the *cyclic operator* or through the bicomplex constructed with the *boundary operator* introduced earlier by Rinehart. These constructions lead to a theory which generalizes de Rham cohomology, S^1-equivariant homology, and group homology in various algebraic settings.

A particularly compelling aspect of cyclic homology is its interaction with K-theory. While algebraic K-groups are notoriously difficult to compute, they admit a canonical map, the *Chern character*, into the periodic cyclic homology which satisfies crucial properties such as *Morita invariance*, *excision*, and *homotopy invariance*. This map preserves much of the rich structure of the K-theory and facilitates computations by passing into a more accessible homological setting. In this sense, cyclic homology serves not only as a tool of intrinsic interest, but also as a crucial bridge between non-commutative algebra and topology.

2 Hochschild (co)homology

The Hochschild homology arose naturally in the algebraic theory as a machine to compute the default of commutativity of an algebra, and more generally of bimodule over it. This homology defines a non-commutative generalization of the notion of differential forms. This analogy is known as the Hochschild-Kostant-Rosenberg theorem and propelled the domain of Non-Commutative Geometry in the early 60's.

There are several possible definitions of Hochschild homology, we will present it in a chronological way by starting with the original construction. This part will be deeply inspired by [Lod92].

2.1 Definition of Hochschild homology

We fix A to be a Fréchet algebra over $\mathbb{C}$. We will note $\otimes := \otimes_\pi$ the projective tensor product over $\mathbb{C}$, which keeps stable the structure of Fréchet algebra (see [Gro54]). Let us denote by $C_n(A) := A^{\otimes n+1}$, the tensor product $n+1$-times of the fixed algebra A. One can define $n+1$ operators $d_i : C_n(A) \to C_{n-1}(A)$, for $0 \le i \le n$, given by:

$$d_i(a_0 \otimes \cdots \otimes a_n) := \begin{cases} a_0a_1 \otimes a_2 \otimes \cdots \otimes a_n & i = 0 \\ a_0 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_n & 0 < i < n \\ a_na_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n \end{cases} \tag{1}$$

One can check that these operators verify the identities:

$$d_id_j = d_{j-1}d_i \text{ for } 0 \le i < j \le n. \tag{2}$$

Then, they induce the differential map b:

$$b := \sum_{i=0}^{n}(-1)^i d_i : C_n(A) \to C_{n-1}(A). \tag{3}$$

verifying the required condition $b \circ b = 0$.

Definition 2.1 *We call **Hochschild homology of** A the homology of the complex $C_\star(A)$ equipped with the differential b:*

$$HH_\star(A) := H_\star(C_\star(A), b).$$

To define the Hochschild homology with coefficients, we want the coefficient space to have a correct structure. As we see in the definition of the d_i's (1), we use the right and the left actions of A on itself, *i.e.* its A-bimodule structure. We recall that a bimodule over A, is a module over the algebra $A^e := A \otimes A^{op}$, where A^{op} is the *opposite side* multiplication algebra with product $a \cdot b := b \cdot a$. If M is a module over A^e (*i.e.* a bimodule over A) one can define the vector space $C_n(A,M) := M \otimes A^{\otimes n}$ and the operators $d_i : C_n(A,M) \to C_{n-1}(A,M)$, for $0 \le i \le n$, whose formulas are the same as (1), with $a_0 \in M$. It induces a differential $b : C_n(A,M) \to C_{n-1}(A,M)$ still given by the alternated sum as in (3).

Definition 2.2 *For any A^e-module M, we call the **Hochschild homology of** A **with coefficients in** M the homology of the complex $C_\star(A,M)$ equipped with the differential b:*

$$HH_\star(A,M) := H_\star(C_\star(A,M), b).$$

The classical Hochschild homology corresponds then to the Hochschild homology with coefficients in A.

The Hochschild homology is compatible with the algebraic structures in any possible ways. If $f : M \to M'$ is a A-bimodule homomorphism, it induces a linear map $f_\star : HH_\star(A,M) \to HH_\star(A,M')$ given by $m \otimes a_1 \otimes \cdots \otimes a_n \mapsto f(m) \otimes a_1 \otimes \cdots \otimes a_n$. Also, any algebra homomorphism $g : A \to A'$ induces a linear map $g_\star : HH_\star(A,M') \to HH_\star(A',M')$, where M' is a A'-bimodule (whose A-bimodule structure naturally arises from g), given by $m \otimes a_1 \otimes \cdots \otimes a_n \mapsto m \otimes g(a_1) \otimes \cdots \otimes g(a_n)$.

2.2 As a derived functor

The Hochschild computes the *default of commutativity* of an algebra, or more generally the default of commutativity between the right and left actions on a bimodule over this algebra. In small degree, $HH_0(A,M)$ corresponds to the cokernel of $b : C_1(A,M) \to C_0(A,M)$ which is given by $b(m \otimes a) = ma - am$, and then to the quotient of M by the space of commutators $[A,M] = \{ma - am \mid a \in A, m \in M\}$:

$$HH_0(A,M) \simeq M/[A,M].$$

For instance, $HH_0(A) = A$ whenever A is commutative.

Lemma 2.1 *If* $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ *is an exact sequence of* A^e*-modules, the following is exact for all* n

$$HH_{n-1}(A,P) \to HH_n(A,M) \to HH_n(A,N) \to HH_n(A,P) \to HH_{n+1}(A,M).$$

This lemma and the computation of HH_0 above tell us that the Hochschild homology is the derived functor of the left-exact functor $M \mapsto M/[A,M] = A \otimes_{A^e} M$, which is:

$$HH_\star(A,M) \simeq \mathrm{Tor}_\star^{A^e}(A,M).$$

This point of view gives a completely new approach to the computation of this homology. Whenever $P_\star(A)$ is a projective resolution of A as a bimodule over itself, one can compute its Hochschild homology as $HH_\star(A,M) = H_\star(M \otimes_{A^e} P_\star(A))$ and the result doesn't depend on the resolution. It gives then different possibilities to compute the same object.

Example(s) Here are some examples of projective resolutions of A as A^e-module.

1. If we choose the projective resolution to be the *bar complex* $C_n^{\mathrm{bar}}(A) := A^{\otimes n+1}$ with the differential map

$$b' = \sum_{i=0}^{n-1} (-1)^i d_i : C_n^{\mathrm{bar}}(A) \to C_{n-1}^{\mathrm{bar}}(A), \tag{4}$$

   we recover the initial definition:

$$HH_\star(A,M) \simeq H_\star\left(M \otimes_{A^e} C_\star^{\mathrm{bar}}(A)\right) = H_\star(C_\star(A,M), b).$$

   The bar complex is known as the *canonical* projective (free) resolution of A as a A^e-module.

2. If A is **commutative** and unital, we can also choose the projective resolution $P_\star(A)$ given by $P_n(A) := A \otimes \bigwedge^n \overline{A} \otimes A$, where $\overline{A} := A/(1 \cdot \mathbb{C})$ is the cokernel of the embedding of the ground field $\mathbb{C}$ in A. The space $P_\star(A)$ is a free graded A^e-module equipped with the differential $\delta := \sum_{i=0}^n (-1)^i \delta_i : P_n(A) \to P_{n-1}(A)$ given by:

$$\delta_i(a \otimes \overline{a_1} \wedge \cdots \wedge \overline{a_n} \otimes a') := \begin{cases} a \otimes \overline{a_2} \wedge \cdots \wedge \overline{a_n} \otimes a' a_1 & i = 0 \\ a \otimes \overline{a_1} \wedge \cdots \wedge \overline{a_i a_{i+1}} \wedge \cdots \wedge \overline{a_n} \otimes a' & 0 < i < n \\ a_n a \otimes \overline{a_1} \wedge \cdots \wedge \overline{a_{n-1}} \otimes a' & i = n \end{cases}$$

In this case, the Hochschild homology can be computed as:

$$HH_\star(A,M) \simeq H_\star(M \otimes_{A^e} P_\star(A)) \simeq M \otimes_A \Omega^\star(A), \tag{5}$$

where $\Omega^\star(A)$ denotes the space of Kähler differentials of A over $\mathbb{C}$ (see [Lod92, §1.3]). Taking for coefficient space $M = A$, we obtain $HH_\star(A) \simeq \Omega^\star(A)$, which is an algebraic analogue of the Hochschild-Kostant-Rosenberg [**?**, HKR]ited in theorem 2.4. This isomorphism is given by the well-defined anti-symetrization map

$$\varepsilon : a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto a_0 \otimes \overline{a_1} \wedge \cdots \wedge \overline{a_n}. \tag{6}$$

2.3 Definition of Hochschild cohomology

Sticking with the point of view of derived functor, we want to make the Hochschild cohomology as a Ext-functor.

DEFINITION 2.3 *For any A^e-module M, we call the* ***Hochschild cohomology of** A **with coefficients in*** *M the homology of the complex $C^\star(A,M) = \mathrm{Hom}_{A^e}(C_\star^{bar}(A), M)$ equipped with the dual differential $b^\vee := (-1)^n(- \circ b') : C^n(A,M) \to C^{n+1}(A,M)$:*

$$HH^\star(A,M) := H^\star(C^\star(A,M), b^\vee).$$

The Hochschild homology without coefficients is defined as $HH^\star(A) := HH^\star(A,A)$.

This cohomology is still compatible with algebraic and bimodule structures as the Hochschild homology. It also computes the *default of commutativity* but in the following way. In small degree, $HH^0(A,M)$ is the kernel of $b^\vee : C^0(A,M) \to C^1(A,M)$ given by $(b^\vee)(m)(a) = ma - am$. It corresponds to the submodule of M where the right and left A-action coincide, which is $Z_A(M)$ the center of M with respect to A:

$$HH^0(A,M) \simeq Z_A(M).$$

When A is commutative, the right and left actions coincide and then $HH^0(A) = A$.

LEMMA 2.2 *If $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ is an exact sequence of A^e-modules, the following is exact for all n*

$$HH^{n+1}(A,M) \to HH^n(A,P) \to HH^n(A,N) \to HH^n(A,M) \to HH^{n-1}(A,P).$$

As before, this long exact sequence and the computation of HH^0 tells us that this cohomology is the derived functor of the right exact functor $M \mapsto Z_A(M) = Hom_{A^e}(A,M)$. In other words:

$$HH^\star(A,M) \simeq \mathrm{Ext}^\star_{A^e}(A,M),$$

as dual statement of the one of the Hochschild homology. Then for any projective resolution $P_\star(A)$ of A as a bimodule over itself, $HH^\star(A,M)$ can be computed as the cohomology groups of $\mathrm{Hom}_{A^e}(P_\star(A), M)$.

2.4 Relation with other (co)homologies

The structure of algebra unifies several other structures as group, manifold, cyclic object, etc. For all of these, it does exists a functor transposing the initial structure to an algebra. For instance, we can send a group to its group algebra, or a manifold to the algebraic space of smooth functions over it, etc. The question of data-loss through these transformation has been studied for decades by comparing (co)homological results.

Let Γ be a discrete group. We denote by $\mathbb{C}[\Gamma]$ its group algebra, which is the space of complex valued Dirac functions on G. The discrete group homology with coefficient in a Γ-module N corresponds to the following Tor-functor:

$$H_\star(\Gamma,N) \simeq \mathrm{Tor}_\star^{\mathbb{C}[\Gamma]}(\Gamma,N).$$

THEOREM 2.3 (Mac-Lane's theorem [ML63])
For all $\mathbb{C}[\Gamma]$-bimodule M, the Hochschild (co)homology of $\mathbb{C}[\Gamma]$ and the group (co)homology of Γ are related in the following sense:

$$HH_\star(\mathbb{C}[\Gamma],M) \simeq H_\star(\Gamma,Ad_\Gamma(M)) \quad and \quad HH^\star(\mathbb{C}[\Gamma],M) \simeq H^\star(\Gamma,Ad_\Gamma(M)),$$

where $Ad_\Gamma(M)$ is the Γ-module for the action $g \cdot m := \delta_g m \delta_g^{-1}$.

This isomorphism is known as the Mac-Lane isomorphism whose proof relies on the fact that the bar complex of $\mathbb{Z}[\Gamma]$ defines a resolution of $\mathbb{Z}$ as the trivial module over Γ.

Let $\mathscr{M}$ be a compact differentiable manifold. One can associate to it the commutative and unital algebra of complex valued functions $\mathscr{C}^\infty(\mathscr{M})$ equipped with the point-wise multiplication. Its Hochschild (co)homology is described as the following.

THEOREM 2.4 (Hochschild-Kostant-Rosenberg's theorem [HKR09])
The Hochschild homology of $\mathscr{C}^\infty(\mathscr{M})$ corresponds to the differential forms over $\mathscr{M}$, while its cohomology corresponds to the exterior algebra $\mathfrak{X}(\mathscr{M})$ of vector fields over $\mathscr{M}$, both through the anti-symmetrization map defined in (6):

$$HH_\star(\mathscr{C}^\infty(\mathscr{M})) \simeq \Omega^\star(\mathscr{M}) \quad and \quad HH^\star(\mathscr{C}^\infty(\mathscr{M})) \simeq \bigwedge^\star \mathfrak{X}(\mathscr{M}).$$

The meaning of this result is that the functor of smooth functions from differentiable manifolds to commutative algebras doesn't loose any cohomological information. This was the main argument that gave rise to the Non-Commutative Geometry and the earlier works of A. Connes and B. Tsygan.

To go further, in a cyclic space framework (see [Lod92, §6]) one can associate to any cyclic space X the cyclic module $\mathbb{C}[X]$, which is the free complex vector space over X. We can compute its Hochschild (co)homology, which takes the following form.

THEOREM 2.5 (Jones' theorem [JP90])
For every cyclic space X, the Hochschild (co)homology of the cyclic module $\mathbb{C}[X]$ is the singular (co)homology of the geometric realization of X:

$$HH_\star(\mathbb{C}[X]) \simeq H_\star(|X|) \quad and \quad HH^\star(\mathbb{C}[X]) \simeq H^\star(|X|).$$

The geometrical realization of a cyclic space naturally carries an action of the circle S^1. The computation of the singular homology of the associated orbit space was a main stake in the 80's, and the response provided can be expressed in terms of cyclic homology.

3 Cyclic (co)homology

In 1963, G. Rinehart proposed a way to compute cyclic homology as a generalization of the De Rham cohomology, fitting with the HKR-theorem. He defined an operator B on the Hochschild complex whose aim is to play the role of the De Rham differential in a non-commutative framework. The juxtaposition of the differentials b and B gives naturally rise to the study of a bicomplex called $\mathscr{B}_\star(A)$ whose total homology is called *cyclic homology* and computes a non-commutative De Rham cohomology.

Almost twenty years after, A. Connes made the following remark. The *cyclic operator* given by a cyclic permutation of the Hochschild complex does commute with the differential b. This operator appears naturally as an algebraic analogue of the circle action on geometric realization of cyclic spaces. Then, the greatest quotient $C_\star^\lambda(A)$ of the Hochschild complex making this cyclic action trivial is a complex once equipped with b. The homology of this complex is called *cyclic homology* and defines an algebraic version of S^1-equivariant singular homology.

In order to stick together these two approaches, A. Connes proposed to study a the well-known bicomplex $CC_\star(A)$ which shows the equivalence between these two definitions and then makes the whole theory relevant. The cyclic homology is simultaneously a non-commutative analogue of the De Rham cohomology and of the S^1-equivariant singular homology. Thanks to the work of J-L. Loday and D. Quillen, it turns out that cyclic homology is also a great tool for the computation of cohomology of matrix Lie algebras and due to the work of A. Connes, it defines a relevant receptacle for the Chern character as well. It is its polyvalence that makes cyclic homology interesting to study.

3.1 Definition of cyclic homology

We fix A to be a complex Fréchet algebra as before. We define the *cyclic operator* $t : C_n(A) \to C_n(A)$ and the *norm operator* $N : C_n(A) \to C_n(A)$ as:

$$t(a_0 \otimes \cdots \otimes a_n) := (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}, \tag{7}$$

$$N := 1 + t + \cdots + t^n. \tag{8}$$

The map t is called *cyclic* as it is given by an action of cyclic groups. These operators commute with d_i defined in (1) in the following way:

$$d_i t = -t d_{i-1} \text{ for } 0 < i \leq n \text{ and } d_0 t = (-1)^n d_n. \tag{9}$$

We obtain the identities $bN = Nb'$ and $b(1-t) = (1-t)b'$, where b' is defined in (4). The following is then well-defined.

DEFINITION 3.1 *We call* ***cyclic homology of*** A *the homology of the total complex of:*

$$CC_\star(A) := \begin{array}{cccccccc} b\downarrow & & -b'\downarrow & & b\downarrow & & -b'\downarrow & \\ A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} \\ b\downarrow & & -b'\downarrow & & b\downarrow & & -b'\downarrow & \\ A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} \\ b\downarrow & & -b'\downarrow & & b\downarrow & & -b'\downarrow & \\ A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} & A & \xleftarrow{N} \end{array}$$

which is $HC_\star(A) := H_\star(\mathrm{Tot}\, CC_\star(A))$.

As for the Hochschild homology, these groups are stable under algebraic transformations. Any algebra homomorphism $f : A \to A'$ induces a linear map $f_\star : HC_\star(A) \to HC_\star(A')$. In small degree, $HC_0(A)$ is the quotient of A by the image of $b : A^{\otimes 2} \to A$ and the image of $1-t : A \to A$. The cyclic operator t is trivial in zeroth degree and then:

$$HC_0(A) = A/[A,A] = HH_0(A).$$

We now try to associate to the cyclic bicomplex the two constructions by G. Rinehart and A. Connes, also computing cyclic homology groups.

3.2 Different points of view

DEFINITION 3.2 *We define the* ***Connes' cyclic complex of*** A *(see [Con85]) as the quotient graded space*

$$C_n^\lambda(A) := A^{\otimes n+1}/(1-t)(A^{\otimes n+1}),$$

equipped with the differential b *as before. The* ***Connes' cyclic homology of*** A *is the homology of the Connes' cyclic complex:*

$$H_\star^\lambda(A) := H_\star(C_\star^\lambda(A), b).$$

One can remark that the Connes' cyclic complex is the quotient of the first column of the cyclic complex through $1-t$, which is the image of the morphism that we call

$$\nu : CC_\star(A) \twoheadrightarrow C_\star^\lambda(A). \tag{10}$$

PROPOSITION 3.1 *For all Fréchet algebra* A*, cyclic homology and Connes' cyclic homology coincide through* ν*:*

$$HC_\star(A) \simeq H_\star^\lambda(A).$$

Remark This result might not happen when the ground field in not contained in $\mathbb{Q}$ because the proof uses the fact that every integer is not a zero-divisor in $\mathbb{C}$.

Proof. It suffices to show that the rows of the bicomplex $CC_\star(A)$ extended with ν are acyclic. We define the following homotopy operators:

$$h' := \frac{1}{n+1} \cdot \mathrm{id} \quad \text{and} \quad h := \frac{-1}{n+1} \sum_{i=1}^{n} i t^i.$$

They verify $h'N+(1-t)h = id$ and $Nh'+h(1-t) = id$ as expected. QED.

Taking the quotient over $1-t$ in the Connes' cyclic complex corresponds topologically to take orbits over an action of S^1. Indeed, the circle S^1 is the geometric realization of the *cyclic space* with two non-degenerated cells, the base point $\star$ in dimension 0 and in dimension 1, the cell corresponding to the cyclic action of $\mathbb{Z}/(n+1)\mathbb{Z}$ on degree n given the cyclic operator t. The cyclic space framework (see [Lod92, §6]) enables us to state the following.

THEOREM 3.2 (*Jones' theorem [JP90]*) *For every cyclic object X, the cyclic homology of its cyclic module $\mathbb{C}[X]$ is the S^1-equivariant singular homology of its geometric realization:*

$$HC_\star(\mathbb{C}[X]) \simeq H_\star(ES^1 \times_{S^1} |X|) = H_\star^{S^1}(|X|).$$

The case $A = \mathbb{C}$ corresponds to the case where the underlying cyclic object $X = \{\star\}$ is a point, which is

$$HC_\star(\mathbb{C}) \simeq H_\star(ES^1 \times_{S^1} \{\star\}) \simeq H_\star(BS^1) \simeq \mathbb{C}[u], \tag{11}$$

where u is of order 2. The previous theorem completes the theorem 2.5 and gives to the cyclic homology a really nice geometrical description.

When A is **unital** we can define the linear map $s : C_n(A) \to C_{n+1}(A)$ given by

$$s(a_0 \otimes \cdots \otimes a_n) := 1 \otimes a_0 \otimes \cdots \otimes a_n. \tag{12}$$

The operators d_i, s and t verify the following relations:

$$d_i s = s d_{i-1} \text{ for } 0 < i < n, d_0 s = id, d_n s = (-1)^n t, \tag{13}$$

$$t^{n+1} = id. \tag{14}$$

These formulas make s verify $b's + sb' = id$, which is that s defines an homotopy of $(C_\star^{\text{bar}}(A), b')$, *i.e.* of the odd columns of $CC_\star(A)$. The operator s is called *extra degeneracy* and leads to the so called *Connes' boundary map* (see [Lod92, §2]):

$$B = (1-t) \circ s \circ N : C_n(A) \longrightarrow C_{n+1}(A). \tag{15}$$

In the case of a unital algebra, the cyclic bicomplex $CC_\star(A)$ is then isomorphic to the one generated by b and B:

$$\mathscr{B}_\star(A) := \begin{array}{ccccccc}
\downarrow b & & \downarrow b & & \downarrow b & & \\
A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A & \xleftarrow{B} & \\
\downarrow b & & \downarrow b & & & & \\
A^{\otimes 2} & \xleftarrow{B} & A & & & & \\
\downarrow b & & & & & & \\
A & & & & & &
\end{array}$$

PROPOSITION 3.3 *For any unital Fréchet algebra A: $HC_\star(A) \simeq H_\star(\text{Tot}\, \mathscr{B}_\star(A))$.*

Through the anti-symmetrization map ε defined in the (6), we know that the Hochschild homology corresponds to the space of differential forms. It turns out this boundary map B, increasing the degree, becomes to the exterior derivative on the De Rham complex, which is:

$$\begin{array}{ccc} A^{\otimes n} & \xrightarrow{B} & A^{\otimes n+1} \\ {\scriptstyle\varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle\varepsilon} \\ \Omega^{n-1}(A) & \xrightarrow{d} & \Omega^{n}(A) \end{array}$$

The bicomplex $\mathscr{B}_\star(A)$ then encapsulated the datum of the differentials forms and of the exterior derivative. The homology of its total complex needs to be related to the De Rham cohomology; that the observation of G. Rinehart in the early 60's. We have the following theorem, which is an extended version of the theorem 2.4.

THEOREM 3.4 *The De Rham cohomology of a compact manifold $\mathscr{M}$ is related to the cyclic homology of its commutative unital algebra of smooth functions $\mathscr{C}^\infty(\mathscr{M})$ by the following:*

$$HC_n(\mathscr{C}^\infty(\mathscr{M})) \simeq \Omega^n(\mathscr{M})/d(\Omega^{n-1}(\mathscr{M})) \oplus H^{n-2}_{dR}(\mathscr{M}) \oplus H^{n-4}_{dR}(\mathscr{M}) \oplus \cdots$$

3.3 Definition of cyclic cohomology

All the operators d_i, t and s can be *dualized* in the sense that on the dual complex $C^\star(A) = \mathrm{Hom}_{A^e}(C^{\mathrm{bar}}_\star(A), A)$ defined in 2.3 we have $d_i^\vee := - \circ d_i$, $t^\vee := - \circ t$ and $s^\vee := - \circ s$. They define of course $b^\vee := \sum_{i=0}^{n}(-1)^i d_i^\vee$, $(b')^\vee := \sum_{i=0}^{n-1}(-1)^i d_i^\vee$ and $N^\vee := 1 + t^\vee + \cdots (^\vee)^n$. It gives rise to the dual theory as follows.

DEFINITION 3.3 *We call* ***cyclic cohomology of*** *A the cohomology of the total complex of:*

$$CC^\star(A) := \begin{array}{ccccccccc} {\scriptstyle b^\vee}\uparrow & & {\scriptstyle -(b')^\vee}\uparrow & & {\scriptstyle b^\vee}\uparrow & & {\scriptstyle -(b')^\vee}\uparrow & & \\ C^2(A) & \xrightarrow{(1-t)^\vee} & C^2(A) & \xrightarrow{N^\vee} & C^2(A) & \xrightarrow{(1-t)^\vee} & C^2(A) & \xrightarrow{N^\vee} & \\ {\scriptstyle b^\vee}\uparrow & & {\scriptstyle -(b')^\vee}\uparrow & & {\scriptstyle b^\vee}\uparrow & & {\scriptstyle -(b')^\vee}\uparrow & & \\ C^1(A) & \xrightarrow{(1-t)^\vee} & C^1(A) & \xrightarrow{N^\vee} & C^1(A) & \xrightarrow{(1-t)^\vee} & C^1(A) & \xrightarrow{N^\vee} & \\ {\scriptstyle b^\vee}\uparrow & & {\scriptstyle -(b')^\vee}\uparrow & & {\scriptstyle b^\vee}\uparrow & & {\scriptstyle -(b')^\vee}\uparrow & & \\ C^0(A) & \xrightarrow{(1-t)^\vee} & C^0(A) & \xrightarrow{N^\vee} & C^0(A) & \xrightarrow{(1-t)^\vee} & C^0(A) & \xrightarrow{N^\vee} & \end{array}$$

which is $HC^\star(A) := H^\star(\mathrm{Tot}\; CC^\star(A))$.

As for cyclic homology, it is functorial for the algebraic structure. In degree zero, $HC^0(A)$ is the intersection of the kernel of $b^\vee : C^0(A) \to C^1(A)$ given by $(b^\vee)(m)(a) = ma - am$ and of $(1-t)^\vee : C^0(A) \to C^0(A)$. The kernel of the first is given by the center of A, denoted $Z(A)$, and the second is the zero map, which gives the analogous result for cyclic cohomology:

$$HC^0(A) = Z(A) = HH^0(A).$$

Of course we recover the same result as in theorem 3.2 but switching indexes by exponents, which means that the cyclic cohomology of the simplicial module associated to a cyclic object is the circle-equivariant singular cohomology of its geometric realization.

3.4 Connes' exact sequence

A way to compute cyclic homology is to used the Connes' exact sequence. It is a long exact sequence involving Hochschild homology groups and cyclic homology groups. Here is the way to think about it.

The first column of the bicomplex $\mathscr{B}_\star(A)$, computing the cyclic homology, is isomorphic to the complex $C_\star(A)$, computing the Hochschild homology. The image of the embedding $I : C_\star(A) \to \mathscr{B}_\star(A)$ is the kernel of the two-degree shift $S : \mathscr{B}_\star(A) \to \mathscr{B}_\star(A)[2]$. The map S called *periodicity map* plays an important role in the theory. The following sequence of complexes is then exact:

$$0 \longrightarrow C_\star(A) \xrightarrow{I} \mathscr{B}_\star(A) \xrightarrow{S} \mathscr{B}_\star(A)[2] \longrightarrow 0.$$

By an argument of a long exact sequence we obtain the following statement.

PROPOSITION 3.5 (*Connes' exact sequence*) *For every unital Fréchet algebra A, the following sequences are exact for all n:*

$$HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A),$$

$$HH^n(A) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \xrightarrow{I} HH^{n+1}(A),$$

where B is the Connes' boundary map defined in equation (15).

The proposition 3.5 can be viewed as a simple algebraic version of the Gysin exact sequence. Indeed, let us give a sphere-bundle, which is a continuous bundle with fiber $F = S^k$:

$$F \longrightarrow E \xrightarrow{\pi} M.$$

The map π induces a pull-back $\pi_\star : H^\star(M) \to H^\star(E)$ in singular homology and a morphism of integration alongs the fibers $\pi^\star : H^\star(E) \to H^{\star-k}(M)$. We call $e(E) \in H^{k+1}(M)$ the Euler class of the bundle. The *Gysin exact sequence* is the following exact sequence for all n (see [BT95]):

$$H^n(E) \xrightarrow{\pi^\star} H^{n-k}(M) \xrightarrow{e(E)\wedge -} H^{n+1}(M) \xrightarrow{\pi_\star} H^{n+1}(E).$$

PROPOSITION 3.6 *For every cyclic space X (see part [Lod92, §6 and §7]), the Gysin exact sequence associated to the sphere-bundle*

$$S^1 \longrightarrow |X| \xrightarrow{\pi} ES^1 \times_{S^1} |X|$$

corresponds to the Connes' exact sequence for the cyclic module $\mathbb{C}[X]$. In other words, the following are isomorphic under the identifications of theorems 2.5 and 3.2:

$$H^n(|X|) \xrightarrow{\pi^\star} H^{n-1}_{S^1}(|X|) \xrightarrow{e(|X|)\wedge -} H^{n+1}_{S^1}(|X|) \xrightarrow{\pi_\star} H^{n+1}(|X|)$$

$$HH^n(\mathbb{C}[X]) \xrightarrow{B} HC^{n-1}(\mathbb{C}[X]) \xrightarrow{S} HC^{n+1}(\mathbb{C}[X]) \xrightarrow{I} HH^{n+1}(\mathbb{C}[X])$$

In particular, this theorem tells us that the periodicity map S can be viewed as a cup product. Indeed, $HC^\star(A)$ defines a module over the algebra $HC^\star(\mathbb{C}) = \mathbb{C}[u]$ where u is the generator element of order 2 as described in (11). It turns out we can describe S as the cup product along u:

$$S : HC^\star(A) \to HC^{\star+2}(A),\ x \mapsto u \cup x.$$

In [Con85], A. Connes defined the suspension map as the two-degree shift map sending a Chern class to the previous one, see (19).

3.5 Decomposition along conjugacy classes

An example of algebra whose cyclic homology would be interested to compute is the group algebra $\mathbb{C}[\Gamma]$ of a discrete group Γ. It is defined as the space of complex valued Dirac functions on Γ, endowed with the convolution product.

We fix an element $x \in \Gamma$ and we write $\langle x\rangle$ for its conjugacy class. For all n, we can define the family spent by:

$$\{\delta_{g_0} \otimes \cdots \otimes \delta_{g_n} \mid g_i \in \Gamma \text{ and } g_0 \times \cdots \times g_n \in \langle x\rangle\} \subseteq \mathbb{C}[\Gamma]^{\otimes n+1}.$$

It is stable under d_i, t and s because these operators are just intern multiplications and permutations. We write $C_\star(\mathbb{C}[\Gamma])_x$ and $CC_\star(\mathbb{C}[\Gamma])_x$ for the subcomplexes of $C_\star(\mathbb{C}[\Gamma])$ and $CC_\star(\mathbb{C}[\Gamma])$ generated by this family.

PROPOSITION 3.7 *The Hochschild complex and cyclic complex of the group algebra* $\mathbb{C}[\Gamma]$ *decompose along the space of conjugacy classes* $\langle G\rangle$ *of* G*:*

$$C_\star(\mathbb{C}[\Gamma]) \simeq \bigoplus_{\langle x\rangle \in \langle G\rangle} C_\star(\mathbb{C}[\Gamma])_x \quad \textit{and} \quad CC_\star(\mathbb{C}[\Gamma]) \simeq \bigoplus_{\langle x\rangle \in \langle G\rangle} CC_\star(\mathbb{C}[\Gamma])_x.$$

One may expect to compute the homology of the *local* complexes to obtain information on the *global* complex. We call $\Gamma_x := \{g \in \Gamma \mid gx = xg\}$ the centralizer of x in Γ. It is equipped with an integral action $\gamma_x : \mathbb{Z} \times \Gamma_x \to \Gamma_x$ given by $\gamma_x(n, g) := x^n g$. Its classifying space $B\Gamma_x$ is then endowed with the circle action $B\gamma_x : S^1 \times B\Gamma_x \to B\Gamma_x$. The singular complex $C_\star(B\Gamma_x)$ and the *local* complex $C_\star(\mathbb{C}[\Gamma])_x$ turns out to be isomorphic via the map (see [Lod92, §7.4.5]):

$$(g_0, \cdots, g_n) \mapsto \delta_{(g_1 \cdots g_n)^{-1} z} \otimes \delta_{g_1} \otimes \cdots \delta_{g_n}.$$

Paired with theorems 2.5 and 3.2, it induces the following identification.

PROPOSITION 3.8 *The Hochschild homology and cyclic homology of the group algebra* $\mathbb{C}[\Gamma]$ *can be computed as singular and* S^1*-equivariant singular homologies, as:*

$$HH_\star(\mathbb{C}[\Gamma]) \simeq \bigoplus_{\langle x\rangle \in \langle G\rangle} H_\star(B\Gamma_x) \quad \textit{and} \quad HC_\star(\mathbb{C}[\Gamma]) \simeq \bigoplus_{\langle x\rangle \in \langle G\rangle} H_\star^{S^1}(B\Gamma_x).$$

We know from classical construction that the singular homology of $B\Gamma_x$ corresponds to the group homology of Γ_x with coefficients in the ground field. Then the Hochschild homology of the group algebra $\mathbb{C}[\Gamma]$ can be expressed in terms of group homology of the centralizers Γ_x. Finally, when x is of infinite order, the space $ES^1 \times_{S^1} B\Gamma_x$ where the circle S^1 acts by $B\gamma_x$ on $B\Gamma_x$ is homotopic to the classifying space of the group $\Gamma_x/(x)$. It gives the following.

THEOREM 3.9 *The Hochschild homology and cyclic homology of the group algebra* $\mathbb{C}[\Gamma]$ *can be computed as follows (see [Lod92, §7.4.10] and [Bur85, Theorem I]):*

$$HH_\star(\mathbb{C}[\Gamma]) \simeq \bigoplus_{\langle x\rangle \in \langle G\rangle} H_\star(\Gamma_x, \mathbb{C}),$$

$$HC_\star(\mathbb{C}[\Gamma]) \simeq \bigoplus_{\langle x\rangle \in \langle G\rangle^{\text{fin}}} H_\star^{S^1}(B\Gamma_x) \oplus \bigoplus_{\langle x\rangle \in \langle G\rangle^{\infty}} H_\star(\Gamma_x/(x), \mathbb{C}),$$

*where the superscripts "*fin*" et "∞" split the cases whether x is of finite or infinite order.*

4 Relation with K-theory

The K-theory is a fundamental tool in non-commutative geometry, a great introduction is given by [Kar78]. Introduced in the 60's by Alexander Grothendieck in order to states the well known Grothendieck-Riemann-Roch theorem, it kept developing to appear as a major tool in the proofs of Bott periodicity and Atiyah-Singer theorems. It is a $\mathbb{Z}/2\mathbb{Z}$-graded theory computing deep algebraic invariants as characteristic classes, Brauer groups, etc. A big issue of this theory is that the computation of its groups is highly non-trivial. The best way to deal with them is to pair K-theory with another theory whose groups are more computable.

Great candidates for the study of K-groups are homology theories verifying the same properties as K-theory: *stable under Morita equivalences*, *stable under smooth homotopies* and *with excision property*. The main issue is that any homology theory respecting these properties is $\mathbb{Z}/2\mathbb{Z}$-graded. Both Hochschild homology and cyclic homology are then irrelevant, but the so called *periodic cyclic homology* verifies the expected properties and defines a $\mathbb{Z}/2\mathbb{Z}$-graded version of these theories. This homology appears to be a generalization of the De Rham cohomology and then defines a great receptacle for the Chern character coming from K-groups. This is what we will present in this section.

4.1 Periodic cyclic homology

We fix A to be a complex **unital** Fréchet algebra as before. We can define the differentials $b : C_n(A) \to C_{n-1}(A)$ and $B : C_n(A) \to C_{n+1}(A)$ on the graded vector space $C_\star(A)$ as in (3) and (15). These operators verify the property:

$$b^2 = B^2 = (b+B)^2 = 0.$$

We splits the Hochschild complex into the *even* and *odd* parts:

$$C_{\text{even}}(A) := \prod_{n\geq 0} C_{2n}(A) \quad \text{and} \quad C_{\text{odd}}(A) := \prod_{n\geq 0} C_{2n+1}(A).$$

The operator $b+B$ sends any even (*resp.* odd) degree element to a pair of odd (*resp.* even) degree elements, and then defines a differential map on the following $\mathbb{Z}/2\mathbb{Z}$-graded complex.

Definition 4.1 *We call* ***periodic cyclic complex of*** A *the following complex:*

$$\widehat{CC}(A) := \ C_{even}(A) \underset{b+B}{\overset{b+B}{\rightleftarrows}} C_{odd}(A)$$

This periodic cyclic complex has only two homology groups. From now we will use the subscript "$\bullet$" for any $\mathbb{Z}/2\mathbb{Z}$-theory in order to distinguish with the classical "$\star$" corresponding to $\mathbb{Z}$-graded theories. Also, we will always write the even part at the left, and the odd part at the right.

Definition 4.2 *We define the* ***periodic cyclic homology of*** A *as the homology of the complex* $\widehat{CC}(A)$*, which is:*

$$HP_\bullet(A) := H_\bullet(\widehat{CC}(A)).$$

The best way to compute periodic cyclic homology is to use the analogue of long exact sequence in homology but in a $\mathbb{Z}/2\mathbb{Z}$-graded framework. It leads to the notion of exact hexagon as follows.

Theorem 4.1 (*Excision property*) *Any admissible extensions of Fréchet algebras* $0 \to A \to B \to C \to 0$ *yields the exact hexagon:*

$$\begin{array}{ccccc} HP_0(A) & \longrightarrow & HP_0(B) & \longrightarrow & HP_0(C) \\ \uparrow & & & & \downarrow \\ HP_1(C) & \longleftarrow & HP_1(B) & \longleftarrow & HP_1(A) \end{array}$$

Theorem 4.2 (*Morita invariance*) *For every Fréchet algebra* $r \geq 1$*, the matrix trace map* $\tau : M_r(A) \to A$ *defines an isomorphism in Hochschild, cyclic and periodic cyclic homology:*

$$HH_\star(M_r(A)) \simeq HH_\star(A),\ HC_\star(M_r(A)) \simeq HC_\star(A) \text{ and } HP_\bullet(M_r(A)) \simeq HP_\bullet(A).$$

The periodic cyclic homology groups can also be computed using filtrations. They play the role of *derived functor property* for the periodic cyclic homology. One can choose any filtration to compute these homology groups, as any projective resolution computes derived functors. A *filtration* F is a family $\{F^nA\}_{n\geq 0}$ of sub-complexes of $\widehat{CC}(A)$ such that $F^{n+1}A \subset F^nA$ and $\bigcup_{n\geq 0} F^nA = \widehat{CC}(A)$. The canonical example of a filtration is the well known Hodge filtration, defined as:

$$F^n_{\text{Hodge}}A := \left(b(C_n(A)) \oplus \bigoplus_{i\geq n} C_i(A), b+B \right) \tag{16}$$

Proposition 4.3 *For any filtration* F*, we can compute the periodic cyclic homology of* A *as:*

$$HP_\bullet(A) \simeq H_\bullet\left(\varprojlim \widehat{CC}(A)/F^nA \right).$$

COROLLARY 4.4 *When the periodicity map S is surjective (which is true for smooth algebras):*

$$HP_\bullet(A) \simeq \varinjlim HC_{\star+2n}(A).$$

If I is a nilpotent ideal of the Fréchet algebra A, then its powers define a filtration of the periodic cyclic complex (see [Lod92, §4.1.15]). If we compute the periodic cyclic homology of A with respect to this filtration, then we obtain the following deep statement.

THEOREM 4.5 (*Goodwillie [Goo86]*) *If I is a nilpotent ideal of A, then*

$$HP_\bullet(A) \simeq HP_\bullet(A/I).$$

This theorem is a weak version of the Goodwillie's theorem in its final form, which states that when I is a nilpotent ideal of A the I-relative K-theory and I-relative cyclic homology of A are rationally isomorphic. A pattern of the proof is given in [Lod92, §11.3.2].

Now one may wonder how to relate periodic cyclic homology with the Hochschild and cyclic theories. Indeed, these can be recovered from the grading complexes $F^nA/F^{n+1}A$ and $\widehat{CC}(A)/F^nA$, in the following way.

PROPOSITION 4.6 *For any filtration F, we have the following identifications whenever n and $\bullet$ are of the same parity:*

$$HH_n(A) \simeq H_\bullet\left(F^nA/F^{n+1}A\right) \quad \textit{and} \quad HC_n(A) \simeq H_\bullet\left(\widehat{CC}(A)/F^nA\right).$$

Proof. We won't prove that this computation is independent of the filtration, but we will show that the result is true for the Hodge filtration F_{Hodge} defined above. Let's arbitrarily say that n is even which is $\bullet = 0$ (the same proof stands for the odd case). When we apply the definitions:

$$H_0\left(F^n_{\text{Hodge}}A/F^{n+1}_{\text{Hodge}}A\right) \simeq H_0\left(C_n(A)/b(C_{n+1}(A)) \underset{b+B}{\overset{b+B}{\rightleftarrows}} b(C_n(A)) \right).$$

Now, the top map B is zero because it increases the degree while the right term is of smaller degree than the left one. Same argument for the bottom maps b which is zero. Finally the bottom B is also zero as b and B anti-commute, so its image is killed. We then obtain:

$$H_0\left(F^n_{\text{Hodge}}A/F^{n+1}_{\text{Hodge}}A\right) \simeq H_0\left(C_n(A)/b(C_{n+1}(A)) \underset{0}{\overset{b}{\rightleftarrows}} b(C_n(A)) \right) \simeq HH_n(A).$$

The cyclic case can be proved similarly. QED.

The main advantage of this point of view is that it unifies Hochschild, cyclic and periodic cyclic homologies in a unique $\mathbb{Z}/2\mathbb{Z}$-graded complex. One can recover some previous statements with this framework. For instance, the sequence of $\mathbb{Z}/2\mathbb{Z}$-graded complexes

$$0 \longrightarrow F^{n-1}A/F^nA \longrightarrow \widehat{CC}(A)/F^nA \longrightarrow \widehat{CC}(A)/F^{n-1}A \longrightarrow 0$$

is exact and induces the Connes' exact sequences as in proposition 3.5. The shift map $S : HC_n(A) \to HC_{n-2}(A)$ appears to be the homological map associated to the surjection $\widehat{CC}(A)/F^nA \to \widehat{CC}(A)/F^{n-2}A$ coming from the axiom $F^nA \subset F^{n-2}A$.

THEOREM 4.7 (*Homotopy invariance*) *If $\mathscr{M}$ is a compact manifold, the periodic cyclic homology of its (unital) algebra $\mathscr{C}^\infty(\mathscr{M})$ of complex smooth functions can be expressed via De Rham cohomology groups through the anti-symmetrization map defined in (6):*

$$HP_0(\mathscr{C}^\infty(\mathscr{M})) \simeq \prod_{n\geq 0} H^{2n}_{dR}(\mathscr{M}) \quad \text{and} \quad HP_1(\mathscr{C}^\infty(\mathscr{M})) \simeq \prod_{n\geq 0} H^{2n+1}_{dR}(\mathscr{M}).$$

This theorem tells us that periodic cyclic homology is stable under smooth homotopies and that there isn't any loss of geometrical data passing from geometry side to algebraic side. We know now that periodic cyclic homology verifies *Excision property*, *Morita invariance* and *Homotopy invariance*. Its computation is then relevant to study K-theory groups through the Chern character that we will present now.

4.2 Chern character

Chern classes are topological invariants which furnish a way to distinguishing two different vector bundles on a smooth manifold. The first class is a complete invariant of line bundles while the top class controls the existence of nowhere vanishing section. They live in even degree cohomology group as they are computed from the curvature of a connection.

The *Chern character* is the map encapsulating all these datum in once, sending a bundle to a polynomial over its Chern classes. This character transforms direct sum and tensor product of vector bundles into sum and cohomological product, and then factorizes over the even K-group of the manifold. By the theorem 4.7 the periodic cyclic homology turns out to be a great receptacle for the Chern character and then a way to study K-group through it.

Let $\mathscr{M}$ be compact manifold and $\pi : E \to \mathscr{M}$ a q-dimensional complex vector bundle over it. Let us fix a *connection* $\omega \in \Omega^1(\mathscr{M}, \mathrm{End}(E))$, *i.e.* a 1-form on the vector bundle $\mathrm{End}(E) := E \otimes E^\star$ with some equivariance properties. The *curvature* of the ω is the 2-form on $\mathscr{M}$ with coefficients in $\mathrm{End}(E)$ given by composing twice the connection $\Omega_\omega := \omega \circ \omega \in \Omega^2(\mathscr{M}, \mathrm{End}(E))$. This curvature computes how close the connection is to be a differential; when it is zero the vector bundle E is said to be *flat*. Its characteristic polynomial $P_\omega(t) = \det(t\Omega_\omega - I)$, can be written degreewise as follows:

$$P_\omega(t) = \sum_{k=0}^{q} c_k(\omega)t^k.$$

The coefficient $c_k(\omega)$ is a sum of cohomological product of k eigenvalues, and then defines an element of $\Omega^{2k}(\mathscr{M})$.

LEMMA 4.8 *For every connections ω and ω' of E:*

- $d(c_k(\omega)) = 0 \in \Omega^{2k+1}(\mathscr{M})$;

- *$c_k(\omega) - c_k(\omega') \in \Omega^{2k}(\mathscr{M})$ is exact.*

In other words, these coefficients define classes in the De Rham cohomology of $\mathscr{M}$ that don't depend on the choice of the connections; we called them **Chern classes of** E and use the notation $c_k(E)$ for all k. They were introduced for the first time in [BH58]. The following map is then well defined:

$$\begin{array}{ccc} \mathrm{VBun}(\mathscr{M}) & \longrightarrow & H^{\mathrm{even}}_{dR}(\mathscr{M}) \\ E & \longmapsto & (c_0(E), c_1(E), \cdots) \end{array}. \tag{17}$$

These characteristic classes are deeply related to Euler class and Pontryagin classes and furnish great topological invariant on vector bundles.

PROPOSITION 4.9 *Here are some topological properties we can extract from the Chern classes:*

- *The first Chern class is a complete invariant of line bundles, which is that if L and L' are two line bundles over $\mathscr{M}$, then $c_1(L) = c_1(L')$ if and only if L and L' are isomorphic.*

- *If $\pi : E \to \mathscr{M}$ is of dimension q, the Chern class $c_q(E)$ is the Poincaré dual of the vanishing space of a generic section s:*

$$c_q(E) = \mathrm{PD}([s^{-1}(0)]).$$

Then, the top Chern class is null when the bundle possesses a nowhere vanishing section.

In the other hand, the exponential of the curvature $\exp(\Omega_\omega) := 1 + \Omega_\omega + \Omega_\omega^2/2 + \Omega_\omega^3/6 + \cdots \in \Omega^{\mathrm{even}}(M, \mathrm{End}(E))$ defines an element of even degree on the De Rham cohomology of $\mathscr{M}$ with coefficient in $\mathrm{End}(E)$. Its trace is the sum of the exponential of the eigenvalues of Ω_ω, and then can be expressed as a polynomial over the Chern classes:

$$\mathrm{tr}(\exp(\Omega_\omega)) = \mathrm{rk}(\Omega_\omega) + c_1 + \frac{1}{2}(c_1^2 + 2c_2) + \frac{1}{6}(c_1^3 + 3c_1c_2 + 3c_3) + \cdots \in \Omega^{\mathrm{even}}(\mathscr{M}),$$

where c_k stands for $c_k(\Omega_\omega)$ and the general term is given by the general Newton polynomial identity. By the previous lemma, we know that the trace $\mathrm{tr}(\exp(\Omega_\omega))$ defines a class in the even degree cohomology of $\mathscr{M}$ which is independent of the choice of connection, we denote it $\mathrm{Ch}(E) := [\mathrm{tr}(\exp(\Omega_\omega))] \in H^{\mathrm{even}}_{dR}(\mathscr{M})$. It verifies the following properties.

THEOREM 4.10 *For every vector bundles E and E' over $\mathscr{M}$:*

- $\mathrm{Ch}(E) = \mathrm{Ch}(E')$ *if E and E' are isomorphic;*

- $\mathrm{Ch}(E \oplus E') = \mathrm{Ch}(E) + \mathrm{Ch}(E')$*;*

- $\mathrm{Ch}(E \otimes E') = \mathrm{Ch}(E) \wedge \mathrm{Ch}(E')$.

In other words, the Chern character factorizes over the Grothendieck group of vector bundles on $\mathscr{M}$, which is called $\mathrm{K}^0(\mathscr{M})$. Also, as seen in theorem 4.7, periodic cyclic homology in zeroth degree encapsulated the even datum of the De Rham cohomology. The following map is then well-defined and called **topological Chern character** defined for the first time in [Hir56]:

$$\mathrm{Ch}: \mathrm{K}^0(\mathscr{M}) \longrightarrow HP_0(\mathscr{C}^\infty(\mathscr{M})). \tag{18}$$

For an algebraic point of view, let us take A to be a unital complex Fréchet algebra and M a module over A as is the first section. The complex $C_\star(A,M)$ becomes a module over the algebra $C_\star(A)$ equipped with the shuffle product. An *algebraic connection of M* is here a map $\omega : C_\star(A,M) \to C_{\star+1}(A,M)$ which verifies the relations:

$$\omega(\alpha\beta) = \omega(\alpha)\beta + (-1)^{\deg(\beta)}\alpha B(\beta),$$

where $\alpha \in C_\star(A,M)$ and $\beta \in C_\star(A)$. Its *curvature* is defined as before as the bicomposition $\Omega_\omega : C_0(A,M) \to C_2(A,M)$. When M is a *finite projective module* of dimension q, which corresponds geometrically to a space of sections over a vector bundle (see [AH12]), the curvature becomes an element of $\Omega_\omega \in C_2(A, End_A(M))$. The coefficients of its characteristic polynomial $P_\omega(t) = \sum_{k=0}^{q} c_k(\omega)t^k$ define some elements $c_k(\omega) \in C_{2k}(A)$ that verify analogous properties.

Lemma 4.11 *For every connections ω and ω' of M:*

- $B(c_k(\omega)) = 0$ *for all* $0 \le k < q$;
- $c_k(\omega) - c_k(\omega') = B(\alpha)$ *where* $\alpha \in C_{2k-1}(A)$,

where B is the Connes' boundary map defined in the equation (15). In particular $c_k(\omega) \in HC_{2k}(A)$ is independent of the choice of connection.

We can define $c_k(M) \in HC_{2k}(A)$ to be the **k-the Chern class** of the finitely projective module M. Also, since the Chern classes of M tive in the kernel of B (except for the top class), they belong to the image of the periodicity map S because of the long exact sequence 3.5. Indeed, we have the following formula for all $0 \le k < q$:

$$c_k(M) = S(c_{k+1}(M)). \tag{19}$$

In [Con85] the suspension map is characterized by this property and turns out to be described as a cup product afterwards. As the periodic cyclic homology is the inductive limit over the cyclic homology groups (see 4.4), we have defined the following map:

$$\begin{array}{ccc} \{\text{f.g.p. modules over } A\} & \longrightarrow & HP_0(A) \\ M & \longmapsto & (c_0(M), c_1(M), \cdots) \end{array}$$

Also, one can view any q-dimensional f.g.p. module M over A as the image of an idempotent matrix $e \in \mathscr{M}_q(A)$. By this analogy, the Chern classes can be expressed as follows:

$$c_k(M) = c_k(e) = \mathrm{tr}\left((-1)^k e^{\otimes 2k+1}\right) \in H^\lambda_{2k}(A),$$

where H^λ is the Connes' cyclic homology defined in 3.2. To obtain a class in the cyclic homology HC, we need to compose with v as defined in (10).

Taking as before the Chern character (independent of the choice of connection) to be $\mathrm{Ch}(M) := [\mathrm{tr}(\exp(\Omega_\omega)] \in HP_0(A)$, we recover the same properties as in theorem 4.10 replacing vector bundles by the f.g.p. modules over A. The even K-group $\mathrm{K}_0(A)$ is defined to be the Grothendieck group of the space of these f.g.p. modules over A. The following map, known as the **algebraic Chern character**:

$$\mathrm{Ch} : \mathrm{K}_0(A) \longrightarrow HP_0(A) \tag{20}$$

stands as a non-commutative analogue of the map (18). It leads to a natural pairing between K-group and periodic cyclic homology, called **Chern-Connes pairing**:

$$\langle -,- \rangle : \mathrm{K}_0(A) \times HP^0(A) \longrightarrow \mathbb{C}, \quad \langle [M], \alpha \rangle := (\alpha \circ \mathrm{Ch})(M).$$

It exists a dual version of this Chern character from *algebraic* K-theory to periodic cyclic cohomology (see [Kar78]):

$$\mathrm{Ch}^\star : \mathrm{K}^0(A) \longrightarrow HP^0(A)$$

which verifies for all f.g.p. module $[M] \in K_0(A)$ and Fredholm module $[\pi] \in K^0(A)$ this integral identity:

$$[\pi] \circ [M] = \mathrm{Ch}^\star(\pi) \circ \mathrm{Ch}(M) \in \mathbb{Z}. \tag{21}$$

Example(s) Here are some examples where this pairing is a striking tool.

1. If $\Gamma = \pi_1(\mathscr{M})$ is the first homotopy group of a $4n$-dimensional manifold $\mathscr{M}$, the Chern-Connes pairing between the signature operator $\mathrm{Ind}_\Gamma D \in K_0(\mathbb{C}[\Gamma])$ and $\alpha \in H^{2n}(\Gamma, \mathbb{C}) \subset HP^0(\mathbb{C}[\Gamma])$ (see 3.9) defines what we call a *higher signature* of $\mathscr{M}$:

   $$\mathrm{sgn}_\alpha(\mathscr{M}) := \langle \mathrm{Ind}_\Gamma D, \alpha \rangle = (\alpha \circ \mathrm{Ch})(\mathrm{Ind}_\Gamma D) \in \mathbb{C}.$$

   The *Novikov's conjecture* stating that these higher signatures are homotopy invariant would be true if we can extend the Chern-Connes pairing from the K-group of $\mathbb{C}[\Gamma]$ to the K-group of $C_r^\star\Gamma$, which is the *rational injectivity* of the assembly map (see [FRR95] for a deeper study).

2. The *Kaplansky-Kadison conjecture* states that if G is a torsion free group, then the only idempotents of its reduced $\mathrm{C}^\star$-algebra $\mathrm{C}_r^\star(G)$ are 0 and 1. We can check that if the canonical trace $\tau : K_0(\mathrm{C}_r^\star(G)) \to \mathbb{C}$, $\tau\left(\sum_{g \in G} a_g g\right) := a_{e_G}$ is integral (takes values in $\mathbb{Z} \subset \mathbb{C}$), then the conjecture is true. In many cases it exists $\alpha \in HP^0(\mathrm{C}_r^\star(G))$ such that the canonical trace map is the result of a Chern-Connes pairing with α:

   $$\tau = \langle -, \alpha \rangle = \alpha \circ \mathrm{Ch}.$$

   Then, the Kaplansky-Kadison conjecture is true when $\alpha = \mathrm{Ch}^\star(\pi)$ is the Chern character of a Fredholm module $[\pi] \in K^0(\mathrm{C}_r^\star(G))$ thanks to identity (21). See [Val02] for a great introduction to K-theory via this conjecture.

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